Networks

Unravelling Complexity

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References

https://www.nature.com/articles/35065725.


https://www.nature.com/articles/30918.
Summary

1. Network Science: An Introduction

2. Dynamics and Topology of Complex Networks

3. Small-World Networks

4. Outlook
Network Science: An Introduction
Kublai reflected on the invisible order that sustains cities, on the rules that decreed how they rise, take shape and prosper, adapting themselves to the seasons, and then how they sadden and fall in ruins. At times he thought he was on the verge of discovering a coherent, harmonious system underlying the infinite deformities and discords (…)

I. Calvino, *Invisible Cities*, ch.8

“The main question was: what is the question?”

A.L. Barabási

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**Structural complexity**

Are there any unifying principles underlying network’s anatomy?

How does the topology affect the network’s properties?

**Dynamical complexity**

How an enormous network of dynamical systems will behave collectively, given their individual dynamics and coupling architecture?
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**Definition**

**Network**

A pair of sets $G = \{P, E\}$, where $P$ is a set of $N$ nodes (or vertices or points) $P_1, \ldots, P_N$ and $E$ is a set of edges (or links or lines) that connect two elements of $P$.

**Examples**

- World Wide Web
- Internet
- Cellular Networks
- Ecological Networks
- Citation Network
- Neural Networks
- (…)

**Figure 1:** Partial map of the Internet (Jan. 15, 2005).
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Methodology

Figure 2: A portion of the molecular interaction map that controls the mammalian cell cycle. [1]

Complications

i. Structural complexity.
ii. Network evolution.
iii. Connection diversity.
iv. Dynamical complexity.
v. Node diversity.
vi. Meta-complications

How do we attack the problem?

Methods of unravelling complexity.

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Dynamics and Topology of Complex Networks
Consider a network of $N$ nodes, each of one is a dynamical system (fixed topology).

\[ \dot{x}_i = f(\bar{x}) , \quad i = 1, \ldots, N \]

If each node has

- **stable fixed points**, the network may display an enormous number of locally stable equilibria.
- a **chaotic attractor**, they can synchronize their fluctuations: *synchronized chaos.* (see Strogatz, S. H., Nonlinear Dynamics and Chaos)
- a **stable limit cycle**, the network of (non-)identical oscillators often synchronize.

**Applications:**
Models from earthquakes to ecosystems, neurons, neutrinos. Most inspired by biology: flashing fireflies, wave propagation in heart, nervous system, brain, intestine (...) [1]
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**Applications:**
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**Algorithm**: take any random two points, connect them with probability $p$.
Random graphs of $N$ nodes and $k$ edges.

**Degree distribution**:

$$P(k) \sim e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}, \quad \langle k \rangle = pN.$$

**Critical phenomena** i.e. giant cluster.

**Goal**: determine at what $p$ a property $Q$ will most likely arise.

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**Figure 3**: (a) Regular Network, (b) Fully Connected Network, (c) Random Graph.

**Figure 4**: Threshold $p \sim N^z$ for various subgraphs.
Algorithm:

1. Growth: Start with $m_0$ nodes, add a new node with $m$ ($\leq m_0$) edges at every $t$.
2. Preferential Attachment: The probability that a new node will be connected to node $i$ is:

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}.$$

Degree distribution:

$$P(k) \sim k^{-\gamma}, \quad \gamma = 3.$$

(Dis-)Advantages: Robustness and resistance to (non-)random attacks. [2]

Figure 5: (d) Scale Free Network.

Popular Scale-Free Networks:
Social Networks, World Wide Web, Internet, Citation Network...
Small-World Networks
Why Small-World?

Small-World topology of networks lie between completely random and regular lattice networks. We can picture these systems like the "Six degree of separation" phenomena.

**Examples**

- The power grid of the western United states.
- The neural network of *C. Elegans*.

**Figure 6**: An example of Small World network with periodic boundary conditions (a ring of nodes).
Construction of the network

The key-concept of Watts Strogatz model of Small-World networks is interpolation.

**Procedure**

- Start from a ring lattice with \( n \) vertices and \( k \) edges.
- Rewire each edge with probability \( p \).

![Image of network models](image)

**Figure 7:** Visual interpretation of the model
Global and local properties

**Characteristic path length, L**
As the name suggests, it is the average distance between two nodes. It’s a *global property*!

**Clustering coefficient, C**
It is a parameter that tells us how much a node is connected on average in the network. It’s a *local property*!

![Figure 8: L,C as a function of p in the Watts-Strogatz model](image)

**These parameters characterize the network**
From a local point of view, the transition to *Small-World* is almost undetectable!
The role of short cuts

Short cut
Increasing $p$ we are forcing the system to find new connections, new short cuts between the nodes. How do they affect the dynamic?

Model of infectious disease
At each step, the infective individuals can infect the neighbours with probability $r$.

The architecture influences the speed and extent of the disease transmission.

Figure 9: $T$ is the time required for global infection and $r_{\text{half}}$ is the probability at which the disease affects half of the population.
This model is a scheme for neurons in the visual cortex

Let a set of $N$ oscillators led by the differential equations:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

For $K > K_C$ a group of oscillators synchronized in phase appears.

Figure 10: Spontaneous synchronization. Time increases from left to right, and from top to bottom.
• Evolutionary models.
• Directed networks, World Wide Web (Broder et al. 2000)
• Weighted networks (local and global optimization)
• Models specific for real networks (far from being random).
Thank you for your attention
Plus Ultra
More on Random Graphs

- **Subgraphs**: The critical probability at which *almost every* graph has a subgraph with \( k \) nodes and \( l \) edges is:
  \[
p_c(N) = \alpha N^{-k/l}.
  \]

- Few results:
  i. \( \langle k \rangle < 1 \): a typical graph is composed of isolated trees.
  ii. \( \langle k \rangle_c = 1 \): Threshold value: the topology changes abruptly.
  iii. \( \langle k \rangle > 1 \): a giant cluster appears: its diameter (i.e. maximal distance between any pair of nodes) is:
  \[
d \propto \frac{\ln(N)}{\ln(\langle k \rangle)}.
  \]
  iv. \( \langle k \rangle \geq \ln(N) \): almost every graph is totally connected.

- **Clustering coefficient**: 
  \[
  C_{\text{rand}} = p = \frac{\langle k \rangle}{N}.
  \]

- Random graphs and statistical physics: **Percolation Theory**.

Figure 11: Bond percolation in 2D.

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Generalized Random Graphs

Random graphs in which any degree distribution is allowed.


Example

Network of the boards of directors of 1000 US companies (bipartite graph).

$p_j, q_k$: probability that a director sits on $j$ boards and that a board consists of $k$ directors, respectively.

$$f_0(x) = \sum_{j=0}^{\infty} p_j x^j, \quad g_0(x) = \sum_{k=0}^{\infty} q_k x^k.$$ 

$r_z$: probability that a random director works with $z$ other co-directors.

$$G_0(x) = f_0 \left( \frac{g_0'(x)}{g_0'(1)} \right),$$

$$r_z = \frac{1}{z!} \frac{d^z G_0}{dx^z} \bigg|_{x=0}.$$
Robustness of real Scale-Free Networks

**Internet and WWW**

- Random errors (e.g. $\sim 0.3\%$ of the routers), hacker attacks.
- High resistance of the giant cluster for random removal of nodes. $f_c^I \sim 0.03$, $f_c^W \sim 0.067$ for attacks.

![Figure 13](image-url)

**Figure 13:** Relative size $S$ of the largest cluster. (a): Internet, $N=6209$. (b): WWW, $N=325729$.

**Ecological Networks**
Solé and Montoya, 2001 [2]

- Human action or environmental changes.
- Parameters:
  1. $S$: Relative size.
  2. $f_{EX}$: Fraction of species becoming isolated due to the removal of other species (secondary extinction).

- Analysis:
  i. **Random removal**: Linear decrease of $S$, $f_{EX} < 0.1$ even for high $f$.
  ii. **Keystone species removal**: $S \sim 0$ for $f \sim 0.2$, $f_{EX} \sim 1$ for low $f$ (e.g. for $f \sim 0.16$ for Silwood Park Web).