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1-D harmonic oscillator in MONDified inertia

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ABSTRACT

In this paper we study the dynamics of a harmonic oscillator with laws of motion prescribed by MOND (Modified Newtonian Dynamics) in its modified inertia formulation. A differential equation for a 1D harmonic oscillator is obtained and several features of its solution are analyzed. Particular attention is given to the deep MOND limit regime, where the equations of motion are significantly different from the Newtonian one.

1. Introduction

MOND is a kind of dynamics proposed in the '80s by Milgrom [1], [2]. This modification of Newtonian dynamics was proposed to fit galaxies rotation curves without using dark matter [3], [4].

A lot of work was done on dynamics of systems subjects to gravity like stars and galaxies [5], [6], [7]. The best prediction of MOND theory concern the physics of galaxies [8], for example the Tully–Fisher and Faber–Jackson relations are in good agreement with the MOND paradigm. On the other hand for cluster of galaxies MOND doesn't explain completely the mass discrepancy.

MOND is fundamentally divided in two formulations: modified gravity (MG) and modified inertia (MI). Modified gravity involves only a modification of the gravitational potential, while modified inertia is a modification of all the forces. So in modified inertia systems subjects to any kind of forces have a modification of their dynamics. MOND was constructed to recover Newtonian dynamics when $a \gg a_0$ where a is the acceleration of the system and a_0 a constant with the dimension of an acceleration. When $a \ll a_0$ the system is in the so called deep MOND limit (DML). This indicates that is the acceleration which discriminates between Newtonian and MOND dynamics. The commonly accepted value of a_0 is $a_0 \approx 1.2 \times 10^{-10}$ m/s^2 and has been obtained by a large amount of physical observations. The most complete and recent survey is [9].

In this paper we want to study a one dimensional harmonic oscillator with MOND dynamics. Obviously a harmonic oscillator is a system which is more reproducible in comparison to galaxy. So if a modification of inertia is really necessary the dynamics of a harmonic oscillator could be a good benchmark.

The modified inertia paradigm is a modification of the Newtonian equation of the form:

$$\vec{F} = m\mu\left(\frac{|\vec{a}|}{a_0}\right)\vec{a} \tag{1}$$

where $\mu(x)$ is called interpolating function. This connects the Newtonian regime to the DML one, $\mu(x)$ is a continuous function. In order to do this interpolation the $\mu(x)$ has to satisfy the following relation:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad |x| \gg 1 \\ x & \text{if} \quad |x| \ll 1. \end{cases} \tag{2}$$

Looking back to the (1) we have that for accelerations much larger than a_0 the Newtonian dynamics is recovered. For accelerations much smaller than a_0 we get the DML. In this limit the force law (1), in one dimension, becomes:

$$F = m\frac{a^2}{a_0}\text{sgn}(a) \tag{3}$$

where sgn is the sign function.

Actually a more general treatment of MI is based on a modification of the kinetic part of the action $S_k[\vec{r}(t), a_0]$. In this contest the kinetic action is a functional of the whole trajectory, and function of the constant a_0. An action of this kind leads to different conserved quantities and adiabatic invariants with respect to MG formulation [7]. It is also possible to construct a theory in MI without *external field effect* (EFE) [10]. We'll talk more about EFE in the discussion of the results. Another fundamental property of such a

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theory is the non locality in time (under the requirement of Galilei invariance) [11], [12].

2. Harmonic oscillator equation

For the harmonic force in 1-D we have:

$$F_h = -kx \tag{4}$$

where k is a positive constant which is related to the angular velocity ω of oscillation by the relation $k = m\omega^2$.

For our purpose of calculation we use the force modification law prescription. So equating (1) and (4) we obtain the differential equation for the harmonic oscillator with modified inertia. For a more general treatment we define some new variables in order to have an adimensional equation. We define: $y \equiv \frac{x}{x_0}$ where x_0 is the maximal amplitude (initial deviation); $\tau \equiv \omega t$ and $\xi \equiv \frac{\omega^2 x_0}{a_0}$. So the equation for the MOND harmonic oscillator reads:

$$\mu(\xi|\ddot{y}|)\ddot{y} = -y. \tag{5}$$

Equation (5) depends on the parameter ξ. Remembering that it is defined as $\xi = \frac{\omega^2 x_0}{a_0}$, it can be thought as a parameter which indicates the average acceleration of the system in units of a_0. Using equation (3), we obtain the general equation for the harmonic oscillator in DML:

$$\xi\ \ddot{y}^2 \mathrm{sgn}(\ddot{y}) = -y. \tag{6}$$

2.1. More on deep MOND limit

Looking at equation (5), we note that it is the argument of the function μ the element which controls the regime of motion. If the argument is much greater than 1 then the equation becomes the same obtained with Newtonian law. While if the argument is much smaller than 1 the equation becomes the (6). Now we want to see when the DML occur. The arguments of μ in eq. (5) are ξ and $|\ddot{y}|$, so there can be two possibilities.

- $\xi \ll 1$, so the typical accelerations of the system are always lower than a_0. Therefore the system is in the DML for all time and also $|\ddot{y}|$ is lower than 1.
- $|\ddot{y}| \ll 1$ but $\xi > 1$. This situation occurs for every system, because there is always, though small, a range of space where the acceleration is lower than a_0. This is easy to check: just look at the Newtonian equation for harmonic oscillator: $\ddot{x} = \omega^2 x$. It's trivial that for enough small x, the acceleration $\ddot{x}$ can be smaller than a_0.

It can be demonstrated that in the DML there exist a whole family of solutions for the equation of motion [13]. This family of solution depends on the particular form of the potential. For the harmonic oscillator the family of solutions has the form:

$$y_\alpha = \alpha^4 y(\tau/\alpha) \tag{7}$$

with α a real parameter.

Now we prove that (7) is actually a solution of equation (6). Let's start by inserting the expression for y_α in (6):

$$\xi\left[\alpha^4 \frac{d^2}{d\tau^2} y\left(\frac{\tau}{\alpha}\right)\right]^2 = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$

$$\xi\left[\alpha^2 \ddot{y}\left(\frac{\tau}{\alpha}\right)\right]^2 = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$

$$\xi\ \alpha^4 \ddot{y}^2\left(\frac{\tau}{\alpha}\right) = -\alpha^4 y\left(\frac{\tau}{\alpha}\right)$$

$$\Rightarrow \xi\ \ddot{y}^2\left(\frac{\tau}{\alpha}\right) = -y\left(\frac{\tau}{\alpha}\right) \tag{8}$$

which is equal to (6). We have supposed $\mathrm{sgn}(\ddot{y}) = 1$, this does not affect the result. For gravitational potential the family of solutions in DML leads to scale invariance for velocity, in accordance with the constant velocity of stars at the edge of the galaxies (more properly when accelerations are lower than a_0). This is not the case for the harmonic oscillator where velocity is not scale invariant as it can be seen easily from equation (7).

3. Analysis and manipulation of equations

Generally equation (5) can not be solved explicitly. The interpolating function makes the differential equation *non linear* unlike the Newtonian one that is linear. This non linearity leads to the following feature: if y and $\tilde{y}$ are solution of (5) then $\tilde{\tilde{y}} = y + \tilde{y}$ is not solution. In other words we cannot superimpose two (or more) different solutions.

We will now look for solutions for the differential equation with initial conditions: $y(0) = 1$ and $\dot{y}(0) = 0$. The condition $y(0) = 1$ is automatically obtained remembering the definition of y: $y = x/x_0$. In fact at the time $t = 0$, x is equal to x_0, so y is equal to one. The second condition has been chosen for simplicity. The Newtonian equation for the harmonic oscillator reads:

$$\ddot{y} = -y \tag{9}$$

with solution:

$$y(\tau) = \cos(\tau) \tag{10}$$

To go further we have to choose a specific interpolating function. Two of the most used function are the *simple interpolating function* and *standard interpolating function*.

$$\mu(x) = \frac{x}{1+x} \text{ simple interpolating function,} \tag{11}$$

$$\mu(x) = \sqrt{\frac{x^2}{1+x^2}} \text{ standard interpolating function.} \tag{12}$$

Using these expressions we can find the associated differential equations. With the *simple interpolating function* (11) in equation (5) we get:

$$\frac{\xi\ \ddot{y}^2\ \mathrm{sgn}(\ddot{y})}{1+\xi|\ddot{y}|} + y = 0. \tag{13}$$

While putting the *standard interpolating function* (12) in (5) we obtain:

$$\sqrt{\frac{\xi^2\ \ddot{y}^2}{1+\xi^2\ddot{y}^2}}\ \ddot{y} + y = 0. \tag{14}$$

Equation (13) is easier to handle. However (13) in that form is not useful. Let's perform some steps to bring (13) in a more clear version. Note that $\mathrm{sgn}(\ddot{y}) = -\mathrm{sgn}(y)$, so (13) becomes:

$$-\frac{\xi\ \ddot{y}^2\ \mathrm{sgn}(y)}{1+\xi|\ddot{y}|} + y = 0. \tag{15}$$

Now for $y > 0$ eq.(15) can be rewritten as:

$$-\xi\ddot{y}^2 - \xi\ddot{y}\ y + y = 0. \tag{16}$$

Basically we have to solve a second degree equation:

$$-\xi x^2 - \xi x\ y + y = 0 \tag{17}$$

where we have replaced $\ddot{y}$ with x. Eq. (17) has the two solutions:

$$x_{1,2} = -\frac{y}{2} \pm \sqrt{\frac{y^2}{4} + \frac{y}{\xi}}. \tag{18}$$

Only the solution with the minus sign has physical meaning. So for $y > 0$ (13) becomes:

$$\ddot{y} = -\frac{y}{2} - \sqrt{\frac{y^2}{4} + \frac{y}{\xi}}. \tag{19}$$

In the same way we can find that for $y < 0$ equation (13) becomes:

$$\ddot{y} = -\frac{y}{2} + \sqrt{\frac{y^2}{4} - \frac{y}{\xi}}. \tag{20}$$

From these two differential equation (19), (20) we can see also that solutions for the modified inertia harmonic oscillator are oscillating functions. Moreover when $y > 0$, we have that $\ddot{y} < 0$ so the solution is concave; while when $y < 0$ we have $\ddot{y} > 0$ so the solution is convex.

3.1. Newtonian limit and deep MOND limit

Let us now start looking at the Newtonian limit. It can be expressed by the formal limit $a_0 \to 0$. The parameter ξ is equal to $\frac{\omega^2 x_0}{a_0}$, so when a_0 goes to 0, $\xi \to \infty$. Taking, for example, equation (19) the term with the denominator ξ goes to zero, and the (19) reduces to:

$$\ddot{y} = -y \tag{21}$$

which is exactly the (9). So Newtonian dynamics is recovered.

Now look for the DML. Start with equation (6), it's easy to check that for $y > 0$ goes to:

$$\ddot{y} = -\sqrt{\frac{y}{\xi}}. \tag{22}$$

While for $y < 0$:

$$\ddot{y} = \sqrt{-\frac{y}{\xi}}. \tag{23}$$

These are in agreement with (19) and (20). Indeed the formal limit for DML is $a_0 \to \infty$, so ξ goes to 0 and the term in (19), (20) with ξ in goes to ∞ and thus is the dominant contribution.

4. Precession of motion

Let us recall the harmonic oscillator equations in MONDified inertia:

$$\ddot{y} = \begin{cases} -\frac{y}{2} - \sqrt{\frac{y^2}{4} + \frac{y}{\xi}} & \text{if} \quad y > 0 \\ -\frac{y}{2} + \sqrt{\frac{y^2}{4} - \frac{y}{\xi}} & \text{if} \quad y < 0 \\ 0 & \text{if} \quad y = 0 \end{cases} \tag{24}$$

Equation (24) can't be resolved explicitly. We know that solution must be a oscillating $C^2(\mathbb{R}_+)$ function. Absolute value of acceleration in modified inertia à la MOND is grater than the Newtonian one: looking at equation (24) we note the terms inside the square roots are greater than $\frac{y}{2}$. So when we add it to the first term of (24) we had a value greater than y. The r.h.s of Newtonian oscillator (9) is exactly y, therefore the MOND acceleration is greater than the Newtonian. This is a first clue to a precession of motion: a stronger acceleration makes oscillations happen in less time. With the word *precession* we mean that the MOND solution get the same value of Newtonian solution but previously in time. Looking at Newtonian solution (10) it's trivial that $T = 2\pi$, with T the period. We want to find the dependence of the period on the ξ of the MOND oscillator. We don't have exact solutions, so we can get only approximated relations. Let us try to describe the system like one with constant translational acceleration and zero initial velocity. The position reads:

$$y = y_0 + \ddot{y}_0 \frac{\tau^2}{2} \tag{25}$$

In our system acceleration isn't constant at all and (25) is a rude approximation. For $\ddot{y}_0$ we use an average acceleration $\ddot{\bar{y}}$ defined as:

$$\ddot{\bar{y}} = \frac{\ddot{y}_{max} - \ddot{y}_{min}}{2}. \tag{26}$$

From equation (24) it's easy to see that $\ddot{y}$ is maximal (in absolute value) when $y = 1$ and minimal when $y = 0$. Replacing these values in (24) and than inserting $\ddot{y}_{max}$ and $\ddot{y}_{min}$ in (26) we get:

$$\ddot{\bar{y}} = -\frac{1}{4} - \sqrt{\frac{1}{16} + \frac{1}{4\xi}}, \tag{27}$$

$\ddot{\bar{y}}$ is calculated for $y > 0$.

Now in equation (25) we use as $\ddot{y}_0$ expression (27), $y_0 = 1$ and $y = 0$, i.e. we look for a $\tilde{\tau} \propto \frac{T}{4}$. Because at a quarter of a period the solution has to go to zero.

$$0 = 1 - \left(\frac{1}{4} + \sqrt{\frac{1}{16} + \frac{1}{4\xi}} \right) \frac{\tilde{\tau}^2}{2}. \tag{28}$$

Form (28) we get:

$$\tilde{\tau} = \frac{2}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\xi}}}}, \tag{29}$$

and so

$$T_{\text{MOND}} \propto \frac{1}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\xi}}}}. \tag{30}$$

Let us call $f(\xi) = \frac{1}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\xi}}}}$, it's easy to check that the Newtonian limit of $f(\xi)$ is: $\lim_{\xi \to \infty} f(\xi) = 1$. To recover Newtonian period $T_N = 2\pi$ we write the "modified" period in this manner:

$$T_{\text{MOND}} \approx T_N \, f(\xi) = \frac{2\pi}{\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{\xi}}}}. \tag{31}$$

4.1. Deep MOND limit period

For the dependence of the period on ξ in particular in the DML, i.e. where MOND dynamics becomes considerable, we can use (22) (or (23)) to find $\ddot{\bar{y}}$ and then with (25), as we did for (30), we get:

$$T_{\text{DML}} \propto \xi^{\frac{1}{4}}. \tag{32}$$

In DML ξ is much lower than 1, so the smaller is ξ, the smaller is T.

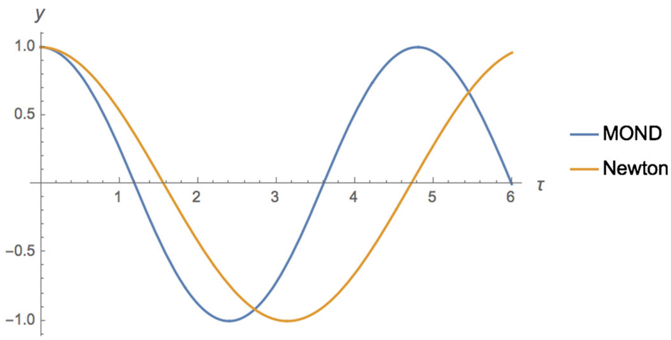

Fig. 1. Comparison between MOND numerical solution with $\xi = 1$, and Newtonian solution.

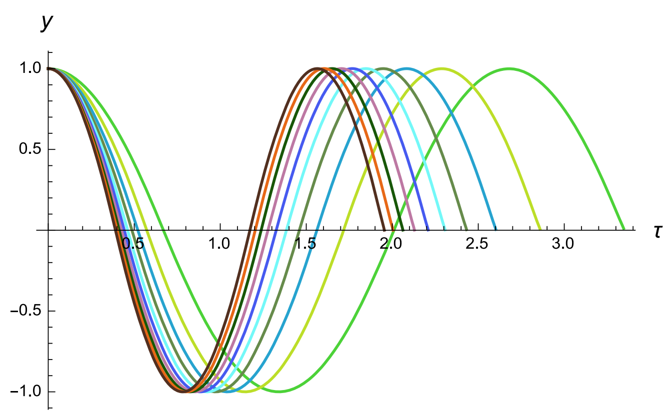

Fig. 2. Ten plots of numerical solutions of MONDified equation for harmonic oscillator. The parameter ξ goes from 1/20 (green line) to 1/200 (black line), with steps of 1/20. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

4.2. Further elements on the solutions of equation (24)

As we said before, an exact solution for equation (24) can't be found. So a first search for solution can be done using a simple code made with *Mathematica*. We do a numerical integration of equation (24) and compare it with the Newtonian solution (see Fig. 1). The expected precession of motion is confirmed.

Looking at Fig. 1 we see that MOND solution is like a cosine function with the argument τ modulated by some factor.

In Fig. 2 we see that when the parameter ξ becomes smaller, also the period becomes smaller.

Let us keep in mind the definition of y: $y = x/x_0$, and of ξ: $\xi = \frac{\omega^2 x_0}{a_0}$. We pay attention in Fig. 2, to get a smaller ξ, having fixed $\omega = \sqrt{k/m}$ and a_0, the only parameter we can move is x_0, i.e. the initial displacement. This is a crucial fact, because the period in MOND depends on the initial displacement x_0 while in Newtonian dynamics doesn't. So when the amplitude becomes smaller also the period gets smaller. Remember that these arguments are valid when the accelerations are lower than a_0. In other words, Newtonian period is scale invariant (with respect to space coordinates), i.e. isocrone oscillations. In MOND oscillations small enough lose isochronism.

Let us try to write an approximated solution for MOND oscillator. Fig. 1 suggests that an good function can be a cosine-like function. We saw that MOND period can be approximated by (31), so we write a "modulation of frequency" (we use the quotation marks because it isn't a real frequency, as it is dimensionless):

$$\Omega_\xi = \frac{2\pi}{T_{\text{MOND}}}. \tag{33}$$

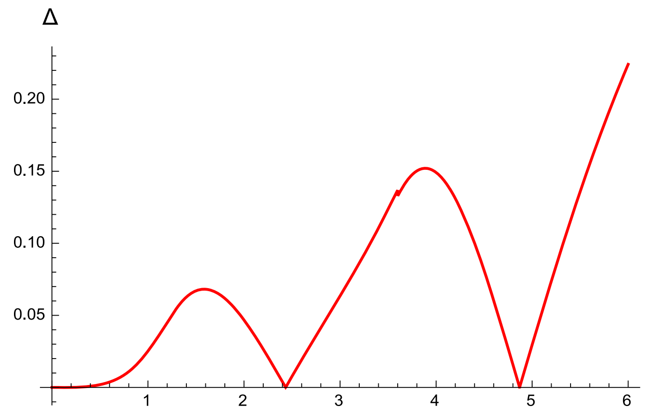

Fig. 3. The function $\Delta(\tau)$ is plotted, the value of ξ is one.

An approximate solution is:

$$y_{\text{MOND}} \approx \cos(\Omega_\xi\ \tau). \tag{34}$$

To check how good (34) is, we look at the difference in absolute value of (34) from the numerical solution $y_{\text{numerical}}$:

$$\Delta(\tau) = |y_{\text{numerical}} - y_{\text{MOND}}|. \tag{35}$$

In Fig. 3 there is a plot of (35).

From Fig. 3 we see that going forward with τ the approximation loses accuracy. This reflect the fact that Ω_ξ does not depends only on ξ but for a real solution it must depends also on time: so $\Omega_\xi(\tau)$. An explicit form for $\Omega_\xi(\tau)$ can not be found, we can only write that a solution of (24) may be of the form:

$$y_{\text{MOND}} = \cos(\Omega_\xi(\tau)\ \tau). \tag{36}$$

This solution can be considered a physical solution, since it can be derived imposing the same two initial condition used for Newtonian case, and it has a unique frequency for given ξ. We have also seen that the period derived in the previous sections has the, correct, Newtonian limit. The same behavior of the frequency of a harmonic oscillator is also pointed out in [7], [12].

5. Experimental conditions and external field effect

In this section we want to discuss the possibility to implement an experiment on Earth which may shows the MOND effects given above. To prove the MOND effects experimentally on Earth is not trivial at all. We have seen that the reference acceleration a_0 is of the order of 10^{-10} m/s^2. These are magnitude that we don't experiment on Earth. The first thing to clarify is: relative to which reference system the acceleration has to be small? If we suppose that it is sufficient to have small acceleration in an arbitrary reference of frame, we end up in an inconsistency of the theory with the usual rules of acceleration addition. So the right *inertial reference frame* (IRF) for a MOND theory must be centered in the center of mass of the galaxy, while the axes must be in the direction of distant quasars. So to perform an experiment we must be at very low acceleration with respect to the IRF [14]. An experiment based on Earth called SHLEM was proposed by Ignatiev [15], where it shows that with an appropriate choice of position and time on Earth it is theoretically possible to test MI on this planet [16].

The complications are not over. The MONDified force formula is non linear in acceleration. The non linearity gives rise to the EFE, it means that the acceleration of a system influence the subsystems. If the acceleration of the bigger system is grater than a_0 the external field effect has the consequence to bring the subsystem (which we suppose with acceleration of the order of a_0) in the Newtonian case.

We said in the introduction that it could be possible to built a MI theory without EFE, but in this case the experiment discussed in [17] helps us. This experiment proved the success of Newton law at very slow acceleration on Earth. This lead to the fact that a consistent theory must embody the EFE.

6. Conclusions

In this paper we have given a first look at the harmonic oscillator using the MOND force expression (modified inertia) instead of the second law of Newton.

The dynamics of galaxies and of the stars which are inside them is not fully known. The most acclaimed hypothesis to solve the issue of rotation curves is dark matter, but its presence has not yet been proven. MOND arises as an alternative theory to solve the dynamical incongruity with respect to Newton theory without using dark matter. To date we do not have a deeper theory which can be put forward. So if a kind of modification of the second law of newton is necessary some real physics must appear also for systems that are not subject to gravity but to any other force.

The study done here shows that if a modified inertia à la MOND is right in slow accelerations regime ($a < a_0$) than there is a precession of the motion with respect to the motion expected with Newtonian dynamics.

In the previous section we have seen that the problems that arise concerning an experiment on Earth are various and non trivial. What we can do is, again, to test the results achieved in an astrophysical contest. We remember that every function, very close to the minimum can be approximated by a harmonic oscillator form (at second order expansion):

$$f(r) = f(r_0) + f'(r_0)(r - r_0) + f''(r_0)(r - r_0)^2 + O(r^3). \tag{37}$$

So a potential $V(r)$ near the minimum can be approximated by a harmonic oscillator.

$$V(r) = V(r_0) + V''(r_0)(r - r_0)^2 \tag{38}$$

An example of this motion is the vertical oscillation of the sun with respect to the galactic plane. As can be seen in the paper [18] the vertical motion of the sun can be modeled by a harmonic oscillator. Today's data are increasingly accurate, so a comparison between the period of oscillation expected using Newtonian dynamics and the MOND one (31) can be done. The fact that some oscillations perpendicular to the disk plane can be tested is also suggested in [7].

Today it is a challenge to find out if a departure from Newton laws of motion may exist. This would be important not only for supporting MOND theory, that could be only an effective theory, but for understanding if a deeper, more fundamental, theory can arise when the accelerations are very low.

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