Fractional calculus and anomalous diffusion

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Between physical phenomena like:

- Anomalous di usion
- Fluid dynamics
- Nuclear physics

and this kind of di erential operator of ordem $\quad 1 < \quad < \quad m$:

$$D \;\; f(t) = \frac{1}{(\;m \qquad)} \int_{0}^{t} \frac{f^{(m)}(\;\;)}{(t \qquad)^{\;+1 \;\; m}} d \qquad + \text{initial conditions terms} \tag{1}$$

On September 30^{th},1695, Guillaume del'Hopital wrote to Leibniz asking him about a particular notation he had used for the n-th derivative

$$\frac{d^n f}{dt^n}$$

His question was: "What would be the result if $n = 1/2$?"
Leibniz's response followed: "An apparent paradox, from which one day useful consequence will be drawn."

Figure: Left: a drop of ink in the water. Right: multiple scattering of light in the foam

Di usion is a process resulting from particle random motion, inducing a net ow of the di usive substance from a region of high concentration to a region of low concentration.
Examples:

- perfume in a room
- neutrons triggering a nuclear reaction
- heat di usion

Second Fick's law and consequences

$$\frac{\partial W(x;t)}{\partial t} = D\frac{\partial^2 W(x;t)}{\partial x^2} \tag{2}$$

$$\langle x^2 \rangle = 2Dt \tag{3}$$

It has been observed in many phenomena, such as

- di usion through percolation and fractal media
- cellular transport
- electron transport in amorphous media

Anomalous di usion has a di erent behaviour of $\langle x^2(t)\rangle$

Figure: Left: 2D Brownian random walk, Right: 1D CTRW. (Sokolov et al., 2012).

Figure: 1D brownian di usion

Master equation:

$$W(x_i; t + \Delta t) = \frac{1}{2} W(x_{i+1}; t) + \frac{1}{2} W(x_{i-1}; t) \tag{4}$$

For $\Delta x; \Delta t \to 0$

$$\frac{\partial W(x;t)}{\partial t} = \frac{\Delta x^2}{2 \Delta t} \frac{\partial^2 W(x;t)}{\partial x^2} \tag{5}$$

By a Laplace-Fourier transform (time and space) with initial condition $W(x;0) = \quad (x)$:

$$sW(k;s) \quad 1 = \quad k^2 D W(k;s) \tag{6}$$

Therefore:

$$W(x;t) = \frac{1}{\sqrt{4 \quad Dt}} \exp \quad \frac{x^2}{2Dt} \tag{7}$$

Brownian case 1D

$$W_j(t + \quad t) = \frac{1}{2} W_{j \quad 1}(t) + \frac{1}{2} W_{j+1}(t) \tag{8}$$

Anomalous case 1D. Fixed length and time step jump probability density function $(x; t) = \quad (x) w(t)$

$$W(k; s) = \frac{1 \quad w(s)}{s} \frac{W_0(k)}{1 \quad (k; s)} \tag{9}$$

Jump length xed. $w(t) \quad A\ (=t)^{+1}$ for long times. $0<\quad 1$.
By inverse Fourier-Laplace transform and Eq. 9,

$$\frac{1}{(1 \quad)}\int_0^t \frac{1}{(t \quad)}\frac{@W}{@}d \quad = \qquad (10)$$

$$D_t\ W = K\ \frac{@^2W}{@x^2}:$$

A straightforward consequence:

$$hx^2(t)i = \frac{2K\ t}{(\quad + 1)} \qquad (11)$$

Cauchy's integral formula:

$$J^{n} f(t) = \frac{1}{(n \quad 1)!} \int_{0}^{t} (t \quad)^{n \quad 1} f(\;) d \; ; \qquad f(t) = 0 \tag{12}$$

becomes:

$$J \; f(t) = \frac{1}{(\;)} \int_{0}^{t} (t \quad) \quad ^{1} f(\;) d \; : \tag{13}$$

$$J \; J \; = J \; J \; = J \; ^{+} \tag{14}$$

Under Laplace Transform

$$D \; f(t) \quad s \; \tilde{f}(s) \quad \sum_{k=0}^{m \quad 1} f^{(k)}(0^{+}) s \quad ^{1 \quad k} \tag{15}$$

The Mittag-Le er function

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, \qquad z \in \mathbb{C}. \qquad (16)$$

$$F = F_A \quad (V \quad u) + F_B$$

$$F_B / \quad D^{\frac{1}{2}}(V \quad u)$$

Figure: A sphere falling in a uid

Let us consider a multilayer generated by in ation rules that act on two consituent layers, say A and B, with di erent refractive indices. In the Fibonacci case: $A \to AB$ and $B \to A$.

Figure: Photonic transport in aperiodic multilayer (Dal Negro, 2017).

For the article here.

Figure: Geant simulations give result di erent from $\frac{dI}{dx} = \quad I$ (Prof.Riggi UniCt.)

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Processes concerning di usion can be improved inducing anomalous di usion

A new old trick: to describe this kind of interactions we may require a change in the (order of) operators

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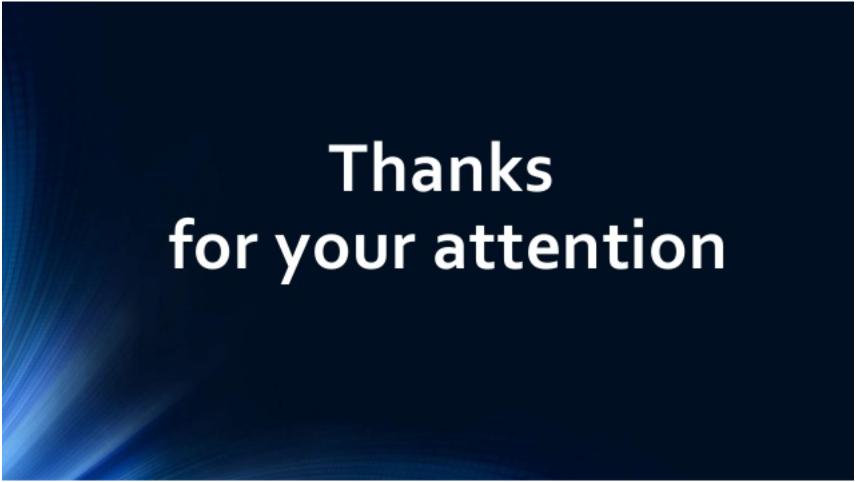
- Processes concerning di usion can be improved inducing anomalous di usion
- A new old trick: to describe this kind of interactions we may require a change in the (order of) operators

The Random's walk guide to anomalous di usion: a fractional dynamics approach. Ralf Metzler, Joseph Klafter

Fractional calculus: integral and di erential equations of fractional order. Goren o, R., Mainardi, F.

Fractional calculus: some basic problems in continuum and statistical mechanics. Mainardi, F.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What Determines the Spreading of a Wave Packet?

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$$k - \text{moment of wavepacket} \propto t^{k\alpha}$$

$$\alpha = \frac{D_2^{\mu}}{D_2^{\psi}}$$

→ Fractality of both spectra and eigenmodes determine pulse spreading

D_2^{μ} – *Correlation Dimension of Local Density of States*

D_2^{ψ} – *Correlation Dimension of Eigenstates of the System*

Connection with fractal kernels and processes