# Networks

## Unravelling Complexity

Vito Dichio, Enrico Drigo

March 27, 2019

Department of Physics, University of Trieste

# References


- Strogatz S. H. , Exploring complex networks. Nature 410, 268-276 (2001). https://www.nature.com/articles/35065725.
- Albert R. , Barabási A. L. , Statistical mechanics of complex networks. Reviews of Modern Physics, Vol. 74, Iss. 1, 47-97 (2002). https://link.aps.org/doi/10.1103/RevModPhys.74.47.
- Watts D. J. , Strogatz S. H. , Collective dynamics of 'small-world' networks. Nature 393, 440–442 (1998). https://www.nature.com/articles/30918.

# Summary

# Network Science: An Introduction

# Motivation and Fundamental Questions

> Kublai reflected on the invisible order that sustains cities, on the rules that decreed how they rise, take shape and prosper, adapting themselves to the seasons, and then how they sadden and fall in ruins. At times he thought he was on the verge of discovering a coherent, harmonious system underlying the infinite deformities and discords (...)
>
> *I. Calvino, Invisible Cities, ch.8*

" The main question was: what is the question? "

*A.L. Barabási*

**Structural complexity**

Are there any unifying principles underlying network's anatomy? How does the topology affect the network's properties?

**Dynamical complexity**

How an enormous network of dynamical systems will behave collectively, given their individual dynamics and coupling architecture?

# Motivation and Fundamental Questions

> Kublai reflected on the invisible order that sustains cities, on the rules that decreed how they rise, take shape and prosper, adapting themselves to the seasons, and then how they sadden and fall in ruins. At times he thought he was on the verge of discovering a coherent, harmonious system underlying the infinite deformities and discords (...)
>
> *I. Calvino, Invisible Cities, ch.8*

“ The main question was: what is the question? ”
*A.L. Barabási*

**Structural complexity**

Are there any unifying principles underlying network's anatomy? How does the topology affect the network's properties?

**Dynamical complexity**

How an enormous network of dynamical systems will behave collectively, given their individual dynamics and coupling architecture?

### Network

A pair of sets $G = \{P, E\}$, where $P$ is a set of $N$ nodes (or vertices or points) $P_1, \ldots, P_N$ and $E$ is a set of edges (or links or lines) that connect two elements of $P$.

### Examples

- World Wide Web
- Internet
- Cellular Networks
- Ecological Networks
- Citation Network
- Neural Networks
- (...)

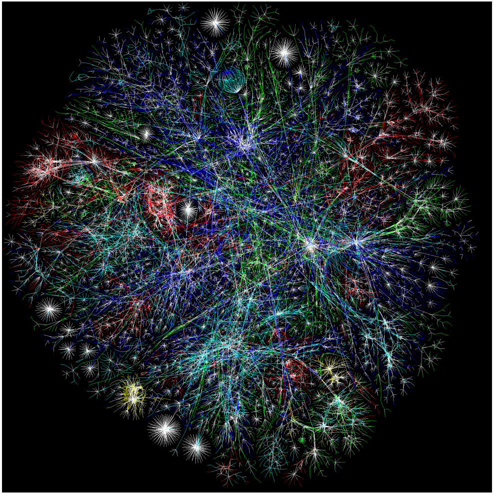


**Figure 1:** Partial map of the Internet (Jan. 15, 2005).

# Definition

**Network**

A pair of sets $G = \{P, E\}$, where $P$ is a set of $N$ nodes (or vertices or points) $P_1, \dots, P_N$ and $E$ is a set of edges (or links or lines) that connect two elements of $P$.

**Examples**

- World Wide Web
- Internet
- Cellular Networks
- Ecological Networks
- Citation Network
- Neural Networks
- (...)

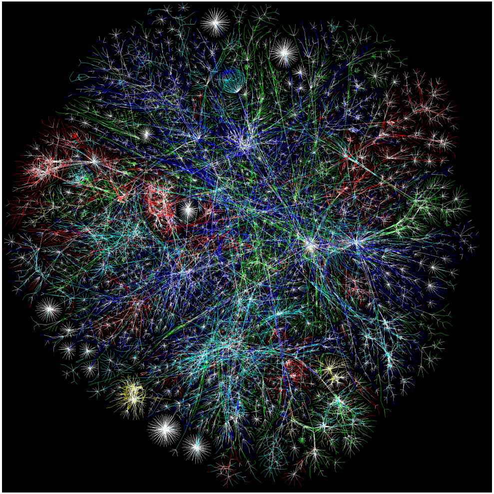


**Figure 1:** Partial map of the Internet (Jan. 15, 2005).

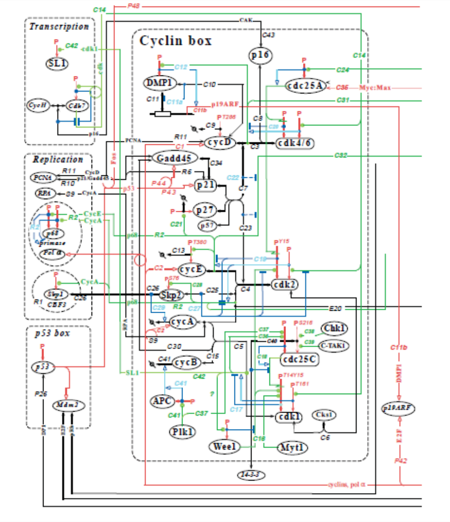



**Figure 2:** A portion of the molecular interaction map that controls the mammalian cell cycle. [1]

## Complications

i. Structural complexity.
ii. Network evolution.
iii. Connection diversity.
iv. Dynamical complexity.
v. Node diversity.
vi. Meta-complications.

How do we attack the problem?
Methods
of
unravelling complexity.

# Methodology

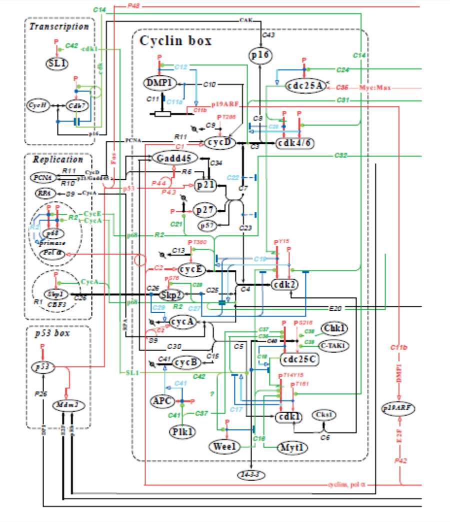



**Figure 2:** A portion of the molecular interaction map that controls the mammalian cell cycle. [1]

## Complications

i. Structural complexity.

ii. Network evolution.

iii. Connection diversity.

iv. Dynamical complexity.

v. Node diversity.

vi. Meta-complications.

---

How do we attack the problem?
Methods
of
UNRAVELLING COMPLEXITY.

# Dynamics and Topology of Complex Networks

Consider a network of N nodes, each of one is a dynamical system (fixed topology).

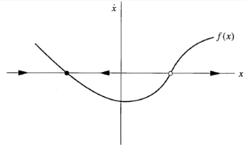
$$\dot{x}_i = f(\bar{x}) \ , \qquad i = 1, \dots, N$$

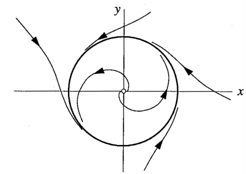
If each node has

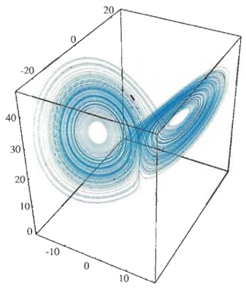
- stable fixed points, the network may display an enormous number of locally stable equilibria.
- a chaotic attractor, they can synchronize their fluctuations: *synchronized chaos*.
  (see Strogatz, S. H., Nonlinear Dynamics and Chaos)
- a stable limit cycle, the network of (non-)identical oscillators often synchronize.

**Applications:**

Models from earthquakes to ecosystems, neurons, neutrinos. Most inspired by biology: flashing fireflies, wave propagation in heart, nervous system, brain, intestine (...) [1]



Consider a network of N nodes, each of one is a dynamical system (fixed topology).

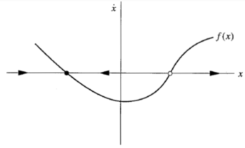
$$\dot{x}_i = f(\bar{x}) \ , \qquad i = 1, \dots, N$$

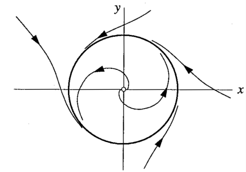
If each node has

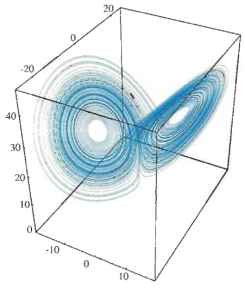
- stable fixed points, the network may display an enormous number of locally stable equilibria.
- a chaotic attractor, they can synchronize their fluctuations: *synchronized chaos*. (see Strogatz, S. H., Nonlinear Dynamics and Chaos)
- a stable limit cycle, the network of (non-)identical oscillators often synchronize.

**Applications:**

Models from earthquakes to ecosystems, neurons, neutrinos. Most inspired by biology: flashing fireflies, wave propagation in heart, nervous system, brain, intestine (...) [1]



# Topology I : Random Graphs

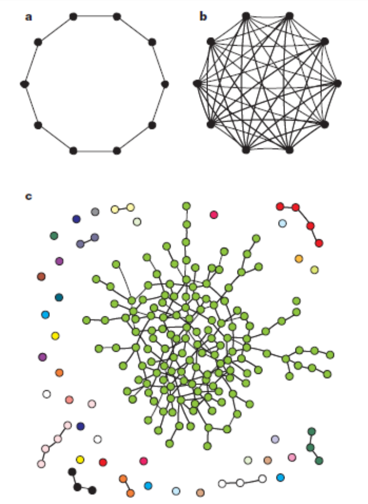



**Figure 3:** (a) Regular Network, (b) Fully Connected Network, (c) Random Graph.

**P. Erdős, A. Rényi** (1959)

- ALGORITHM: take any random two points, connect them with probability $p$. Random graphs of $N$ nodes and $k$ edges.
- DEGREE DISTRIBUTION:

$$P(k) \sim e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}, \quad \langle k \rangle = pN.$$

- CRITICAL PHENOMENA i.e. giant cluster.
- GOAL: determine at what $p$ a property $Q$ will most likely arise.

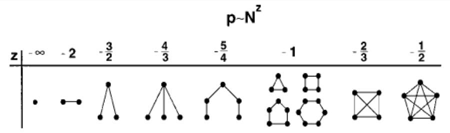



**Figure 4:** Threshold $p \sim N^z$ for various subgraphs.

**A. L. Barabási, R. Albert** (1999)
`http://networksciencebook.com/`

- ALGORITHM:
  1. Growth : Start with $m_0$ nodes, add a new node with $m$ ($\leq m_0$) edges at every $t$.
  2. Preferential Attachment : The probability that a new node will be connected to node $i$ is:

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} .$$

- DEGREE DISTRIBUTION:

$$P(k) \sim k^{-\gamma} , \quad \gamma = 3.$$

- (DIS-)ADVANTAGES: Robustness and resistance to (non-)random attacks. [2]

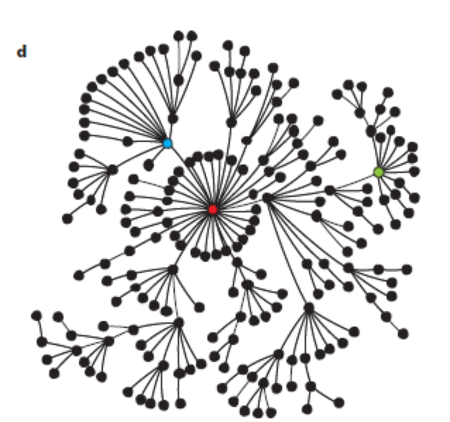



**Figure 5:** (d) Scale Free Network.

**Popular Scale-Free Networks:**
Social Networks, World Wide Web, Internet, Citation Network...

# Small-World Networks

Small-World topology of networks lie between completely random and regular lattice networks.
We can picture these systems like the "Six degree of separation" phenomena.

**Examples**

- The power grid of the western United states.
- The neural network of *C. Elegans*.

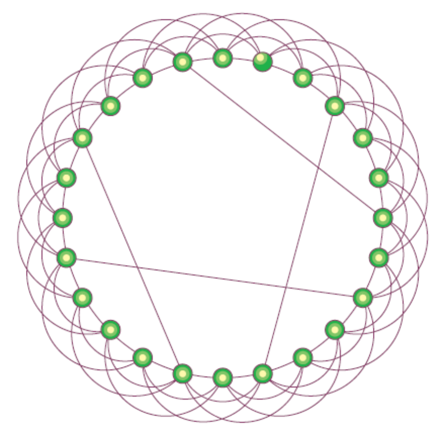


**Figure 6:** An example of Small World network with periodic boundary conditions (a ring of nodes).

The key-concept of *Watts Strogatz* model of Small-Wolrld networks is *interpolation*.

**Procedure**

- Start from a ring lattice with $n$ vertices and $k$ edges.
- Rewire each edge with probability $p$.

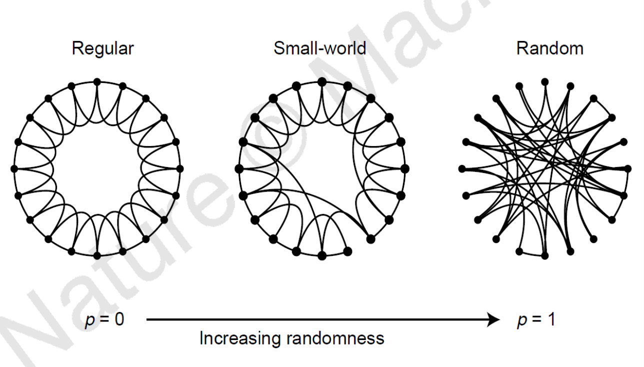



**Figure 7:** Visual interpretation of the model

# Global and local properties

**Characteristic path length , L**

As the name suggest, it is the average distance between two nodes.
It's a *global property*!

**Clustering coefficient , C**

It is a parameter that tells us how much a node is connected on average in the network.
It's a *local property*!

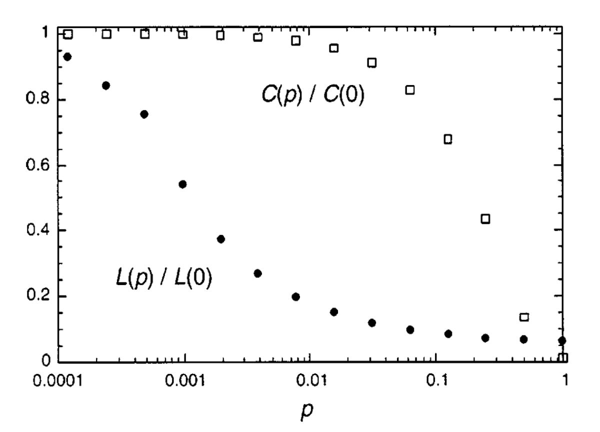



**Figure 8:** L,C as a function of p in the Watts-Strogatz model

These parameters characterize the network

From a local point of view, the transition to *Small-World* is almost undetectable!

## Short cut

Increasing $p$ we are forcing the system to find new connections, new short cuts between the nodes. How do they affect the dynamic?

## Model of infectious disease

At each step, the infective individuals can infect the neighbours with probability $r$.

THE ARCHITECTURE INFLUENCES THE SPEED AND EXTENT OF THE DISEASE TRASMISSION.

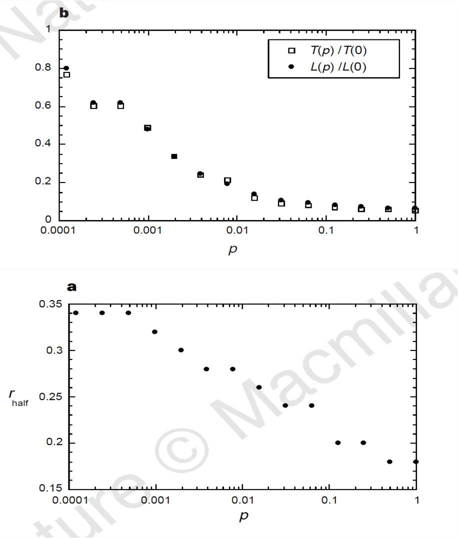



**Figure 9:** $T$ is the time required for global infection and $r_{half}$ is the probability at which the disease affects half of the population.

# Coupled phase oscillators with small-world connectivity

**This model is a scheme for neurons in the visual cortex**

Let a set of N oscillators led by the differential equations:

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin\left(\theta_j - \theta_i\right)$$

For $K > K_C$ a group of oscillators synchronized in phase appears.

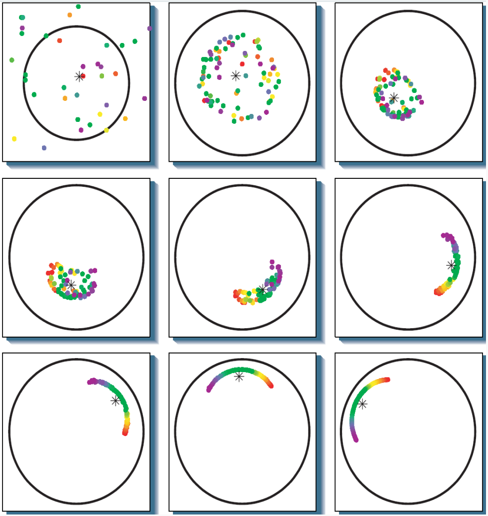
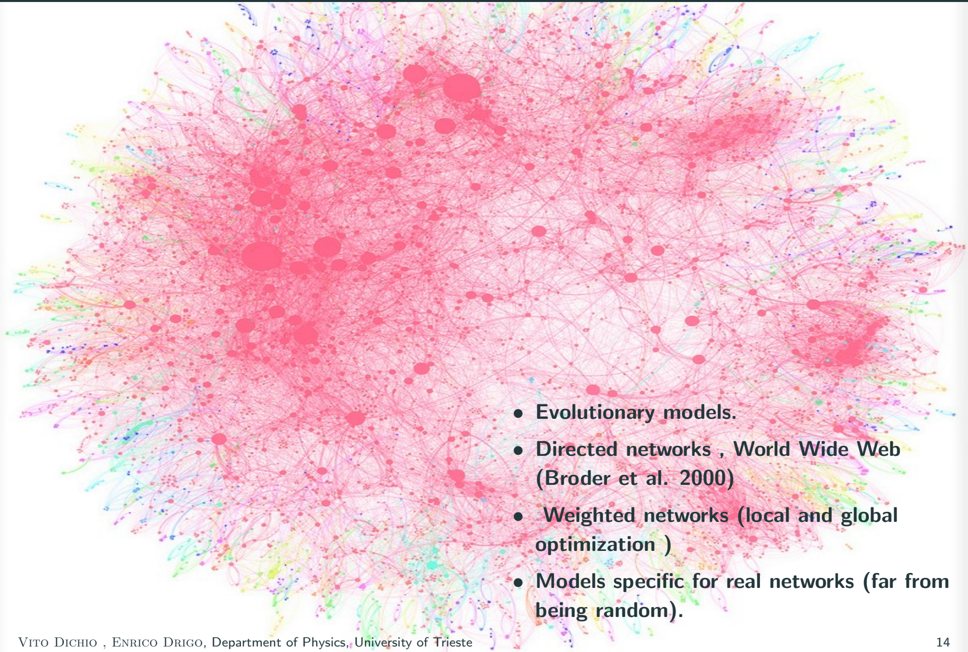


**Figure 10:** Spontaneous sychronization. Time increases from left to right, and from top to bottom.

# Outlook

# What is left



- Evolutionary models.
- Directed networks , World Wide Web (Broder et al. 2000)
- Weighted networks (local and global optimization )
- Models specific for real networks (far from being random).

Thank you

for your attention

# Plus Ultra

# More on Random Graphs

- SUBGRAPHS: The critical probability at which *almost every* graph has a subgraph with $k$ nodes and $l$ edges is:

$$p_c(N) = \alpha \; N^{-k/l} \; .$$

- Few results:
  i. $\langle \mathbf{k} \rangle < \mathbf{1}$ : a typical graph is composed of isolated trees.
  ii. $\langle \mathbf{k} \rangle_{\mathbf{c}} = \mathbf{1}$ : Threshold value: the topology changes abruptly.
  iii. $\langle \mathbf{k} \rangle > \mathbf{1}$ : a giant cluster appears: its diameter (i.e. maximal distance between any pair of nodes) is:

$$d \propto \frac{ln(N)}{ln(\langle k \rangle)} \; .$$

  iv. $\langle \mathbf{k} \rangle \geqslant \mathbf{ln}(\mathrm{N})$ : almost every graph is totally connected.

- CLUSTERING COEFFICIENT:

$$C_{rand} = p = \frac{\langle k \rangle}{N} \; .$$

- Random graphs and statistical physics: PERCOLATION THEORY.

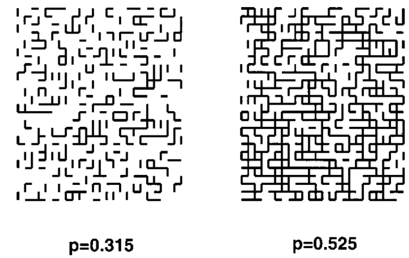



**Figure 11:** Bond percolation in 2D.

Random graphs in which any degree distribution is allowed.

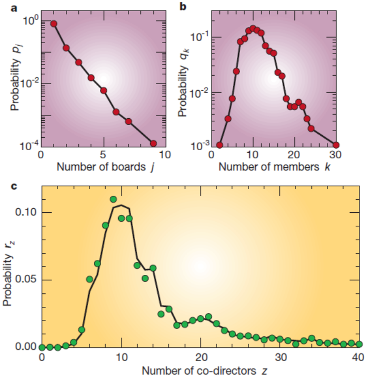



**Figure 12:** (a): Boards per director. (b): Directors per board. (c): Director's total number of co-directors.

**Newman, Strogatz and Watts**, 2000 [1]

GENERATING FUNCTIONS APPROACH.

**Example**

Network of the boards of directors of 1000 US companies (BIPARTITE GRAPH).

$p_j$, $q_k$: probability that a director sits on j boards and that a board consists of k directors, respectively.

$$f_0(x) = \sum_{j=0}^{\infty} p_j x^j \ , \quad g_0(x) = \sum_{k=0}^{\infty} q_k x^k \ .$$

$r_z$: probability that a random director works with $z$ other co-directors.

$$G_0(x) = f_0\left(\frac{g_0'(x)}{g_0'(1)}\right) \ ,$$

$$r_z = \frac{1}{z!} \frac{d^z G_0}{dx^z}\bigg|_{x=0} \ .$$

# Robustness of real Scale-Free Networks

## Internet and WWW

**Albert, Jeong and Barabási**, 1999, 2000 [2]

- Random errors (e.g. $\sim 0.3\%$ of the routers), hacker attacks.
- High resistance of the giant cluster for random removal of nodes. $f_c^I \sim 0.03$ , $f_c^W \sim 0.067$ for attacks.

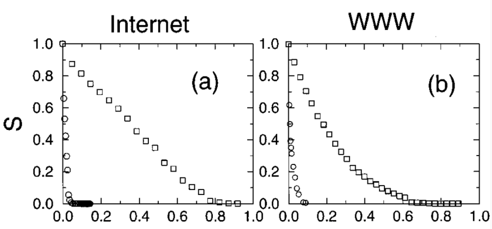



**Figure 13:** Relative size $S$ of the largest cluster. (a): Internet, N=6209. (b): WWW, N=325729.

## Ecological Networks

**Solé and Montoya**, 2001 [2]

- Human action or environmental changes.
- Parameters:
  1. **S**: Relative size.
  2. $\mathbf{f_{EX}}$: Fraction of species becoming isolated due to the removal of other species (secondary extinction).
- Analysis:
  i. **Random removal**: Linear decrease of $S$, $f_{EX} < 0.1$ even for high $f$.
  ii. **Keystone species removal**: $S \sim 0$ for $f \sim 0.2$, $f_{EX} \sim 1$ for low $f$ (e.g. for $f \sim 0.16$ for Silwood Park Web).